ELSEVIER Journal of Pure and Applied Algebra 110 (1996) 315–320

JOURNAL OF PURE AND APPLIED ALGEBRA

C-normality and solvability of groups[☆]

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Communicated by K.W. Gruenberg; received 3 February 1994

Abstract

A subgroup H is called c-normal in group G if there exists a normal subgroup N and G such that $HN = G$ and $H \cap N \leq H_G$ where $H_G =: Core(H) = \bigcap_{g \in G} H^g$ is the maximal normal subgroup of G which is contained in H. We obtain some results about the c-normal subgroups and the solvability of groups.

1. Introduction

The relationship between the properties of maximal subgroups of a finite group G and the structure of G have been studied by many people. It is well known that a finite groups G is nilpotent if and only if every maximal subgroup of G is normal in G. As for the class of supersolvable groups, Huppert's well-known theorem shows that a finite group G is supersolvable if and only if every maximal subgroup of G has a prime index in G. In terms of normality, we have that G is supersolvable if and only if every maximal subgroup of G is weakly normal in G [8], Theorem 1.8.7. Also, some people try to characterize group structure using as few maximal subgroups as possible [2–4, 7]

Definition 1.1. Let G be a group. We call a subgroup H c-normal in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$.

It is clear that a normal subgroup of G is a c-normal subgroup of G but the converse is not true. For example, $S_3 = C_3 \rtimes C_2, C_2 \ntrianglelefteq S_3$ but C_2 is c-normal in S_3.

Definition 1.2. We call a group G c-simple if G has no c-normal subgroup except the identity group 1 and G.

☆ Project supported in part by National Natural Science Foundation of China and STFG.

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SSDI 0022-4049(94)00090-6

We can easily show that G is c-simple if and only if G is simple, which is useful in our discussion.

In this paper, we give some analogue properties of normal subgroups for c-normal subgroups. We prove that a finite group G is solvable if and only if every maximal subgroup of G is a c-normal in G. We also try to minimize the number of the maximal subgroups to characterize the structure of G.

Let p be a prime and p' the complementary set of primes. Let G be a finite group. Then we denote $M \lessdot G$ to indicate that M is a maximal subgroups of G. Also, $|G:M|_p$ denotes the p-part of $|G:M|$. Consider the following families of subgroups.

Definition 1.3. We define
$\mathscr{F} = \{M\colon M \lessdot G\}$.
$\mathscr{F} = \{M\colon M \lessdot G\}$ with $|G:M|$ is composite.
$\mathscr{F}_p = \{M\colon M \lessdot G, |G:M|_p = 1\}$.
$\mathscr{F}_{pc} = \mathscr{F}_p \cap \mathscr{F}_c$.
$\mathscr{F}^p = \{M\colon M \lessdot G,\ N_G(P) \leq M\}$ for a $P \in Syl_p(G)$.
$\mathscr{F}^s = \bigcup_{p \in \pi(G)} \mathscr{F}^p$.
$\mathscr{F}^{pc} = \mathscr{F}^p \cap \mathscr{F}_c$.
$\mathscr{F}^{sc} = \mathscr{F}^s \cap \mathscr{F}_c$.

Definition 1.4.
$\Phi_p(G) = \bigcap\{M\colon M \in \mathscr{F}_p\}$ if $\mathscr{F}_p$ is non-empty; otherwise $\Phi_p(G) = G$.
$S_p(G) = \bigcap\{M\colon M \in \mathscr{F}_{pc}\}$ if $\mathscr{F}_{pc}$ is non-empty; otherwise $S_p(G) = G$.
$\Phi^p(G) = \bigcap\{M\colon M \in \mathscr{F}^p\}$ if $\mathscr{F}_p$ is non-empty; otherwise $\Phi^p(G) = G$.
$\Phi^s(G) = \bigcap\{M\colon M \in \mathscr{F}_s\}$ if $\mathscr{F}_s$ is non-empty; otherwise $\Phi^s(G) = G$.
$S^p(G) = \bigcap\{M\colon M \in \mathscr{F}^{pc}\}$ if $\mathscr{F}^{pc}$ is non-empty; otherwise $S^p(G) = G$.
$S^s(G) = \bigcap\{M\colon M \in \mathscr{F}^{sc}\}$ if $\mathscr{F}^{sc}$ is non-empty; otherwise $S^s(G) = G$.

It is clear that all the above subgroups are characteristic subgroups of G.

2 Preliminaries

2.1 Basic properties

Lemma 2.1. *Let G be a group. Then*

(1) *If H is normal in G, then H is c-normal in G;*

(2) *G is c-simple if and only if G is simple;*

(3) *If H is c-normal in G, $H \leq K \leq G$, then H is c-normal in K;*

(4) *Let $K \trianglelefteq G$ and $K \leq H$. Then H is c-normal in G if and only if H/K is c-normal in G/K.*

Proof. (1) $HG = G$ and $H \cap G = H \trianglelefteq G$; hence H is c-normal in G.

(2) By (1), we only need to prove the part "if". Assume that G is simplest but G is not c-simple. Then there exists a non-trivial subgroup H, $1 < H < G$ such that H is c-normal in G. By definition, there exists $N \trianglelefteq G$ such that $H N = G$, which yields that $N \neq 1$ and so $N = G$. It follows that $1 \neq H = H \cap G \leq H_G \lhd G$, contrary to our assumption.

(3) $HN = G$, $K = K \cap G = H(K \cap N)$. $K \cap N$ is normal in K and $H \cap N \cap K \leq H_G \cap K \leq H_K$.

(4) Suppose that H/K is c-normal in G/K. Then there exists $N/K \trianglelefteq G/K$ such that $G/K = (H/K)(N/K)$ with $(H/K) \cap (N/K) \leq (H/K)_{G/K}$. It is easy to see that $G = H\ N$ and $H \cap N \leq H_G$. The converse is the same. □

Lemma 2.2. *Let G be a finite group. Then*

(1) *$\Phi^p(G)$ is p-closed for every $p \in \pi(G)$;*

(2) *$\Phi^s(G)$ is nilpotent;*

(3) *$S^p(G)$ is p-closed for the maximal prime divisor $p \in \pi(S^p(G))$;*

(4) *$S^s(G)$ has Sylow tower.*

Proof. (1) Let $P_1 \in Syl_p(\Phi^p(G))$. By Sylow's theorem, there exists $P \in Syl_p(G)$ such that $P_1 = P \cap \Phi^p(G)$. If $P_1 \ntrianglelefteq G$, then there exists $M \lessdot G$ such that $N_G(P) \leq N_G(P_1) \leq M \lessdot G$ and so $\Phi^p(G) \leq M \in \mathscr{F}^p$. By Frattini argument, $G = \Phi^p(G)N_G(P_1) \leq M \lessdot G$, a contradiction. Therefore $P_1 \trianglelefteq G$.

(2) It is clear that $\Phi^s(G) = \bigcap_{p \in \pi(G)} \Phi^p(G)$. By (1), $\Phi^s(G)$ is p-closed for every $p \in \pi(G)$, this follows that $\Phi^s(G)$ is nilpotent.

(3) Let $P_1 \in Syl_p(S^p(G))$. By Sylow's theorem, there exists $P \in Syl_p(G)$ such that $P_1 = P \cap S^p(G)$. If $P_1 \ntrianglelefteq G$, then there exists $M \lessdot G$ such that $N_G(P) \leq N_G(P_1) \leq M \lessdot G$. By the Frattini argument, $G = S^p(G)N_G(P_1)$. If $|G:M| = q$ is a prime, by Sylow's theorem, $q = 1 + kp$. But $q \mid |S^p(G)|$ and hence $q < p$, a contradiction. Hence $|G:M|$ is composite and $M \in \mathscr{F}^{ps}$. This yields that $G = S^p(G)N_G(P_1) \leq M \lessdot G$, a contradiction; therefore $P_1 \trianglelefteq G$.

(4) Let p be the maximal prime divisor or $|S^s(G)|$. The same argument as (3) shows that $S^s(G)$ is p-closed. Let $P \in Syl_p(S^s(G))$. Then P char $S^s(G)$ char G and it is easy to show that $S^s(G/P) = S^s(G)/P$. By induction, $S^s(G/P)$ has a Sylow tower and so does $S^s(G)$. □

Lemma 2.3. *Let G be a finite group. Then*

(*a*) *G is nilpotent if and only if $G = \Phi^s(G)$.*

(*b*) *G is nilpotent if and only if M is normal in G for every $M \in \mathscr{F}^s$.*

(c) *G is nilpotent if and only if G/N is nilpotent for a normal subgroup N of G which is contained in $\Phi^s(G)$.*

Proof. (a) $G = \Phi^s(G)$ if and only if $\mathscr{F}^s = \emptyset$ if and only if $N_G(P) = G$ for every Sylow p-subgroup of G and for every prime $p \in \pi(G)$ if and only if G is nilpotent.

(b) By Frattini argument and (a).

(c) Suppose that G/N is nilpotent and M be a maximal subgroup of G and $M \in \mathscr{F}^s$. Then $M/N \trianglelefteq G/N$ by (a) and hence $M \trianglelefteq G$. It follows from (b) that G is nilpotent. □

Lemma 2.4. *Let G be a finite group. Then*

(a) G is supersolvable if and only if $|G; M|$ is a prime for every $M \in \mathscr{F}^s$.

(b) G is supersolvable if and only if $= S^s(G)$. G is nilpotent if and only if M is normal in G for every $M \in \mathscr{F}^s$.

(c) G is supersolvable if and only if G/N is supersolvable for a normal subgroup N of G which is contained in $S^s(G)$.

Proof. (a) By Huppert's well-known theorem, we only need to prove the part "if". Let p be the largest prime of $\pi(G)$ and $P \in Syl_p(G)$. Then $P \trianglelefteq G$. In fact, if it is false, then there exists a maximal subgroup M of G with $N_G(P) \leq M \lessdot G$. By assumption, $|G:M| = q$ for a prime $q < p$, which yields that G/M_G is isomorphic to a subgroup of the symmetric group S_q and hence $|G/M_G| \,|\, q!$. In particular, $P \leq M_G$. The Frattini argument yields that $G = M_G N_G(P) \leq M$, a contradiction. It is easy to show that G/N satisfies the hypotheses of G for every minimal normal subgroup N and G. Suppose that (a) is false and let G be a minimal counterexample. Then G has unique minimal normal subgroup N. It is easy to prove that $N = P = F(G)$ and $G = N \rtimes M$ with $M \lessdot G$. Let q be the largest prime of $\pi(M)$. Since M is supersolvable, we have that $M \leq N_M(Q) \leq N_G(Q)$ for $Q \in Syl_q(M) \cap Syl_q(G)$. Since M is a maximal subgroup of G and $Q \ntrianglelefteq G$ it follows that $M = N_G(Q)$. By our assumption, $|N| = |G:M| = p$ is a prime, which yields that G is supersolvable, contrary to our choice.

(b) By (a), G is supersolvable if and only if $\mathscr{F}^{sc} = \emptyset$, that is if and only if $G = S^s(G)$.

(c) The same argument as Lemma 2.3(c). □

3. Theorems

Theorem 3.1. *Let G be a finite group. Then G is solvable if and only if every maximal subgroup of G is c-normal in G.*

Proof. Suppose that every maximal subgroup M and G is c-normal in G. We prove that G is solvable. Assume that it is false and let G be a minimal counterexample. If G is simple, the by Lemma 2.1(2), G is c-simple, it follows that $M = 1$ and G is a group of prime order, a contradiction. Hence, we assume that G is not simple. It is clear that the hypotheses of the theorem are satisfied by any quotient group G/K of G. A trivial argument shows that G has unique minimal normal subgroup K with $K \nleq \Phi(G)$. Then there exists a maximal subgroup $M \lessdot G$ such that $K \nleq M$, i.e. $G = KM$. Since M is c-normal in G, there exists $N \trianglelefteq G$ such that $G = MN$ and $N \cap M \leq M_G = 1$. Since $1 \neq N$ it follows that $K \leq N$ and so $K \cap M = 1$. Hence $|N| = |G:M| = |K|$, $K = N$. For any maximal subgroup $L \lessdot G$ with $L_G = 1$, we have $KL = G$. Since L is c-normal in G, the same argument shows that $|G:L| = |K|$. By a result of Baer [1, Lemma 3], K is solvable. It is clear that G/K satisfies the hypotheses of G. The minimal choice of G implies that G/K is solvable. Now that both K and G/K are solvable follows that G is solvable, a contradiction.

Conversely, suppose that G is solvable and $M \lessdot G$. If $M_G \neq 1$, consider G/M_G and use induction on $|G|$, we get M/M_G is c-normal in G/M_G. From Lemma 2.1 it follows that M is c-normal in G. Assume $M_G = 1$. Let N be a minimal normal subgroup of G which is certainly abelian. Then $G = n\,M$ and $N \cap M \leq M_G = 1$. By definition, M is c-normal in G. □

In the direction of limiting the number of maximal subgroups which we control, we prove the following result.

Theorem 3.2. *Let G be a finite group. Then G is solvable if and only if there exists a solvable c-normal maximal subgroup M of G.*

Proof. Assume the theorem is false and let G be a minimal counterexample. Let M be a c-normal solvable maximal subgroup of G. Then G must satisfy the following:

(a) M is core-free. If $M_G \neq 1$, then M/M_G is a solvable c-normal maximal subgroup of G/M_G, which yields that G/M_G is solvable and hence G is solvable, a contradiction.

(b) There exists a minimal normal subgroup K of G such that $G = K \rtimes M$. Since M is c-normal in G, there exists a normal subgroup N and G such that $G = NM$ and $M \cap N \leq M_G = 1$. Let L be a minimal normal subgroup of M, which is certainly abelian p-subgroup with $p \in \pi(M)$.

(c) $(p, |K|) = 1$ and $C_K(L) = 1$. In fact, $C_K(L)$ is normalized by both M and K and hence $C_K(L) \trianglelefteq G$. If $C_K(L) = K$, then $1 \neq L \leq M_G$, contrary to (a). Therefore $C_K(L) = 1$. The orbit formula implies that $(p, |K|) = 1$.

(d) K is a q-subgroup for a prime q.

By [5, Theorem 6.2.2] and (c), there exists an unique L-invariant Sylow q-subgroup Q of K for every prime $q \in \pi(K)$. For any element $m \in M$, $(Q^m)^L = (Q^L)^m = Q^m$, i.e. Q^m is also a L-invariant q-Sylow subgroup of K. From the uniqueness it follows that $Q^m = Q$ and hence Q is M-invariant. Since M is a maximal subgroup of G, $Q \rtimes M = G = K \rtimes M$. This yields that $K = Q$ is a $q =$ subgroup.

Now both K and G/K are solvable implies that G is solvable, contrary to our choice. □

We can also discuss p-solvability in terms of c-normality.

Theorem 3.3 *Let G be a finite group and p be the maximal prime divisor of $|G|$. If M is c-normal in G for every non-nilpotent maximal subgroup $M \in \mathscr{F}^{pc}$, then G is p-solvable.*

Proof. Assume that the theorem is false and G is a minimal counterexample. Then

(1) $\mathscr{F}^{pc} \neq \emptyset$. If $\mathscr{F}^{pc} = \emptyset$, then $G = S^p(G)$ is p-closed by Lemma 2.2(3). Hence, $P \trianglelefteq G$ for Sylow p-subgroup P and G is p-solvable, a contradiction.

(2) M is c-normal in G for every $M \in \mathscr{F}^{pc}$. It is sufficient to prove that G has no nilpotent maximal subgroup M with $M \in \mathscr{F}^{pc}$. In fact, suppose that there exists $M \in \mathscr{F}^{pc}$ with M nilpotent. Since G is non-solvable, Thompson's theorem [5, 10.3.2] implies that $M_2 \neq 1$. If M is a 2-subgroup, then $p = 2$ and G is a 2-group, contrary to our choice. Hence, G is a non-solvable and $M_{2'} \neq 1 \neq M_2$. By [6, Theorem 1], $M_{2'}$ is normal in G. It is easy to show that $G/M_{2'}$ satisfies the hypotheses of G. The minimal choice of G yields that $G/M_{2'}$ is p-solvable. Now $M_{2'}$ is solvable implies that G is p-solvable, a contradiction.

(3) G has an unique minimal normal subgroup N and G/N is p-solvable. By (1) and Lemma 2.1(2), G is not simple. For every non-trivial normal subgroup N of G, the minimal choice of G yields that G/N is p-solvable. Since p-solvable groups form a saturated formation, there exists an unique minimal normal subgroup N of G.

If $p \nmid |N|$ or $|N|$ is a p-group, then N is p-solvable and then G is p-solvable, contrary to our choice. We assume that $p \mid |N|$ and $N = N_p$. The Frattini argument yields that $G = N_G(N_p)$. Let P be a Sylow p-subgroup of G such that $N_p = P \cap N$. Since $1 \neq N_p \neq N$, $N_G(N_p) \neq G$. There exists $M \lessdot G$ such that $N_G(P) \leq N_G(N_p) \leq M$; Hence of $M \in \mathscr{F}^{p}$. From $N \nleq M$ it follows that $M_G = 1$. If $|G:M| = q$ with q a prime, then $q < p$ and $|G| \mid q!$, a contradiction. Hence $|G:M|$ is composite and $M \in \mathscr{F}^{pc}$. By (2), M is a c-normal in G and it follows that there exists a normal subgroup K such that $N \cap M \leq K \cap M \leq M_G = 1$. $|G:M|_p = 1$ yields that $|N|_p = 1$, a contradiction. There is no counterexample. □

Acknowledgements

The author would like to thank the New York Group Theory Cooperation and the CCNY for their hospitality and he would also like to thank the referee for several comments.

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